Lecture 11: Finish Alexpoly + obstr. from Donaldson's thm

- sorry for cancelling class
- plan for rest of semester:

Today: finish some sm. obstructions owed from before

Next week: two special topics – exotic $\mathbb{R}^4$s
Jul 4 – high dims e.g. $\Sigma_g \hookrightarrow S^4$.

Last lecture: - Open problems / other directions
Jul 11 - Question session

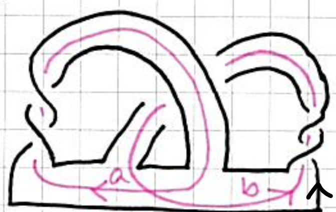
$\longrightarrow$

One last invariant of knots:

Def$^{\underline{n}}$: $K \subseteq S^3$ with Seifert surface F of genus g,

with "symplectic basis" $\{a_i, b_i\}_{i=1}^{g}$

F



i.e. $a_i \pitchfork b_j = \begin{cases} pt & i=j \\ \emptyset & o/w \end{cases}$

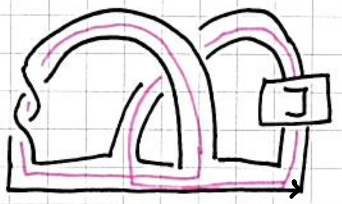
$a_i \pitchfork a_j = b_i \pitchfork b_j = \emptyset \quad \forall i \neq j$

$Arf(K) = (-1)\cdot(1) = 1$

[sometimes the "geom. real$^{\underline{n}}$" of a symp. basis]

$$Arf(K) := \sum_{i=1}^{g} lk(a_i, a_i^+)\, lk(b_i, b_i^+) \mod 2 \in \mathbb{Z}/2.$$

E.g. $Wh(J)$



$Arf(Wh(J)) = (-1)\cdot(0) = 0 \quad \forall J$

Fact: $\Delta_K(t) = 1 \Rightarrow K$ alg. slice $\Rightarrow Arf(K) = 0$

Indeed, $Arf(K) = 0 \iff \Delta_K(-1) \equiv \pm 1 \mod 8$

$Arf(K) = 1 \iff \Delta_K(-1) \equiv \pm 3 \mod 8$

More generally, Arf defined for quadratic forms over fields of char 2.

[forms, applying to "quadratic refinement" of the Seifert pairing]

Goal: $\Delta_K(t) = 1 \Rightarrow K$ TOP slice [Converse not true!]

Recall from previous lecture [Lecture 7, May 16]

compact, oriented. "Equivariant intersection & self-intersection numbers"

$\lambda: H_2(W; \mathbb{Z}[\pi_1 W]) \times H_2(W; \mathbb{Z}[\pi_1 W]) \to \mathbb{Z}[\pi_1 M]$

$\mu: H_2(W; \mathbb{Z}[\pi_1 W]) \to \mathbb{Z}[\pi_1 W]/_{g \sim g^{-1}}$

via Hurewicz: $H_2(M;\mathbb{Z}[\pi_1 M]) \cong \pi_2(M)$.

0-surgery characterisation of sliceness.

$K \subseteq S^3$ TOP slice iff $M_K := S^3_0(K) = \partial W^4$ for W compact, oriented s.t. (i) $\mathbb{Z} \cong H_1(M_K) \longrightarrow H_1(W)$ is an isom.

(ii) $\pi_1 W$ is normally gen. by μ_K (meridian) $\subseteq M_K$.

(iii) $H_2 W = 0$

Sphere embedding thm [Freedman - Quinn '90]

$f, g : S^2 \looparrowright W^4$ s.t. $\mu(f) = 0$, $\lambda(f,g) = 1$, g has trivial normal bundle.

$\pi_1 W$ = abelian or finite

Then f is homotopic to a loc. flat emb. $\bar{f}$

& g is homotopic to an immersion $\bar{g}$ s.t. $\bar{f} \pitchfork \bar{g} = pt$.

[$\exists$ more general theorems]

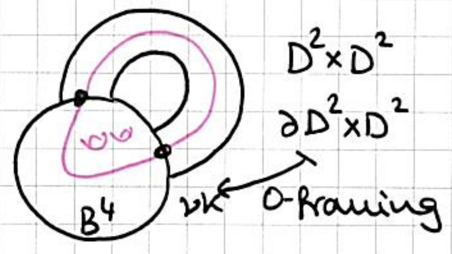
$\longrightarrow$

Proof sketch [rougher idea of surgery thy approach to building mfld

Goal: Build W s.t. $\partial W = M_K$, $W \simeq S^1$.

Step 1: Build V^4 s.t. $\partial V = M_K$ s.t. V spin i.e. $w_1(V) = 0 = w_2(V)$

equiv. tangent bundle trivial over 2-skeleton



$X_0(K)$

$F :=$ Seifert surface, genus g

$F^{\uparrow} :=$ Seifert surface F, interior pushed in radially

$\hat{F} := F^{\uparrow} \cup D^2 \times 0$
"core"

Note: $\partial X_0(K)$

Define $V := (X_0(K) \setminus \nu\hat{F}) \cup (H_g \times S^1)$

genus g handlebody.

Fact: $\mathrm{Arf}(K) = 0 \Rightarrow V$ spin

Compute: $\pi_1(V) \cong \mathbb{Z} \langle\langle \mu_K \subseteq M_K \rangle\rangle$.

$H_1(M_K; \mathbb{Z}[\mathbb{Z}]) = 0 \Rightarrow$ equivariant int form on V is nonsingular

i.e. Alex poly = 1

Consider $(\pi_2(V), \lambda, \mu) \in L_4(\mathbb{Z}[\mathbb{Z}])$
the L-group of "non-singular quadratic forms",
i.e. λ sesquilinear, Hermitian, unsing. bilinear form
μ quadratic form.

Considered modulo "hyperbolic forms" ie. of the form $\oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

fact: $L_4(\mathbb{Z}[\mathbb{Z}])$ well-understood
$\cong 8\mathbb{Z}$, generated by "E_8 form" [see part 2]

fact: $\exists$ closed, TOP 4-mfld E with $(\pi_2 E, \lambda_E, \mu_E) = E_8$ form.

Step 2: Construct $V' := V \# \pm n E$ s.t. $(\pi_2 V', \lambda_{V'}, \mu_{V'}) = 0 \in L_4(\mathbb{Z}[\mathbb{Z}])$
fact: still spin

Step 3: Construct W using surgery.

$(\pi_2 V', \lambda_{V'}, \mu_{V'}) = 0 \in L_4(\mathbb{Z}[\mathbb{Z}]) \Longrightarrow$ int form is $\oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In the nicest case: $\exists f, g : S^2 \looparrowright V'$ s.t. $\lambda(f,g) = 1$
V' spin $\Longrightarrow$ f, g trivial normal bundle,
$\mu(f) = 0 = \mu(g)$

Sphere embedding thm $\Rightarrow \exists \bar{f}, \bar{g}$ s.t. $\bar{f} : S^2 \hookrightarrow V'$
$\bar{g} : S^2 \looparrowright V'$ $\quad \bar{f} \pitchfork \bar{g} = pt.$

Let $W := V' \setminus (\bar{f} \times D^2) \cup (D^3 \times S^1)$

Check: $\pi_2 W = 0$, $\pi_1 W \cong \mathbb{Z}$, and

$$\begin{tikzcd} M_K \arrow[r] \arrow[d, "\partial"'] & S^1 \\ W \arrow[ur, "\simeq"'] & \end{tikzcd}$$

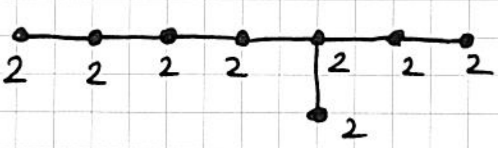
$\square$

Corollary: Wh(RHT) is TOP slice.

Part II: Obstructions from Donaldson's theorem.

E8 form : $\mathbb{Z}^8 \times \mathbb{Z}^8 \longrightarrow \mathbb{Z}$

adjacency matrix for:



$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix} =: \text{E8 matrix}$$

Check: $\det(E8) = 1$

$E8(v,v) > 0 \quad \forall v \neq 0$ i.e. E8 positive definite.

[Note: many people rather use E8 to mean the negative definite version]

$\longrightarrow$

Recall: M^4 closed, oriented, by Poincaré duality

$\exists$ intersection form $Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$

$[A] \quad [B] \quad A \pitchfork B$

emb. surfaces.

i.e. a symmetric, unimodular, integral, bilinear form.

Q: Which ↑ forms are realised as intersection forms of 4-mflds?

A1: [Freedman] Every such form is realised for a TOP 4-mfld

e.g. the E8 form.

[Can do construction at end if someone asks]

A2: [Rochlin] E8 is not the int form of a $\pi_1 = 1$ smooth 4-mfld

[More generally, $\sigma(Q_M)$ for M closed, smooth, spin is div by 16]

A3: [Donaldson] M^4 closed, smooth, oriented, Q_M positive definite.

Then $Q_M \cong n\langle +1 \rangle$ for some n.

[These smooth exclusions are roughly also why we know surgery thy "does not work" in dim 4 smoothly]

Pos. def. forms, even:

rk	8	16	24	32	40
#	1	2	4	$>10^7$	$\gg 10^{51}$

← none of these are rep sm'ly.

<u>Idea</u>: K smoothly slice $\Rightarrow \Sigma_2(K)$, double branched cov

$= \partial V^4$ s.t. $H_*(V;\mathbb{Q}) = 0$.

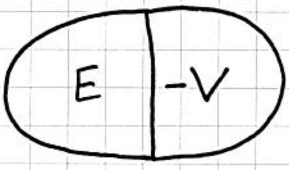
V compact, oriented

[Similarly, $S^3_{\pm 1}(K) = \partial W^4$ s.t. W compact, oriented and $H_*(W;\mathbb{Z}) = 0$]

Choose K s.t. $\Sigma_2(K) = \partial E$ [or $S^3_{\pm 1}(K) = \partial E$]

s.t. E compact, oriented, & $E \cup_{\Sigma_2(K)} -V$ [or $E \cup_{S^3_{\pm 1}(K)} -W$]

has <u>nonstandard</u> pos. definite intersection form.

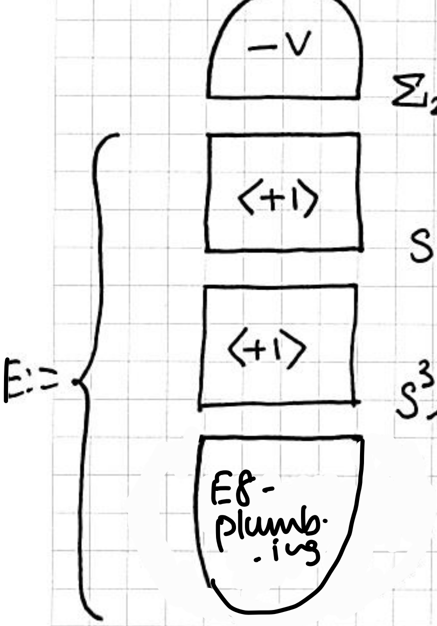


$Q_{E \cup V} \not\cong n\langle +1\rangle \Rightarrow\Leftarrow$

$\Rightarrow$ K not sm. slice. $\square$

[The precise details of the following are not important. Focus on the proof outline above!]

[following Akbulut (unpublished) Cochran-Gompf '88]



$\Sigma_2(\mathrm{Wh}(RHT)) \cong S^3_{\frac{1}{2}}(RHT \#_r RHT)$

$RHT \#_r RHT \rightsquigarrow RHT$ single + to − cr. change

$S^3_{1/2}(RHT)$

$S^3_{\frac{1}{n}}(K) \geq S^3_{\frac{1}{n-1}}(K) \quad \forall K$

$S^3_1(RHT)$ = Poincaré homology sphere

$Q_{E \cup -V} \cong E8 \oplus 2\langle +1\rangle \not\cong 10\langle +1\rangle \Rightarrow\Leftarrow$

$\Rightarrow$ Wh(RHT) <u>not</u> sm slice.